Homework 9 (Due 4/02/2014)

Math 622

March 30, 2014

1. A submartingale is defined on page 74, of the text. Jensen's inequality is stated on page 18 and the conditional Jensen inequality is stated on page 70.

a) Show: if $X(t)$, $t \geq 0$, is a martingale, and if f is a convex function such that $E[|f(X(t))|] < \infty$ for all t, then $f(X(t))$ is a submartingale.

b) Let S denote an asset price process. If the risk-free interest rate in a risk-neutral model is $r = 0$, and h is a convex payoff function, show that $h(S(t))$ is a submartingale. Show that the value of an American and European option expiring at T are the same, if the payoff function is h.

2. Let r be the risk-free rate, and consider a risk-neutral model,

$$\begin{aligned} dS_1(t) &= rS_1(t)\,dt + \sigma_1 S_1(t)\,d\widetilde{W}_1(t) \\ dS_2(t) &= rS_2(t)\,dt + \sigma_2 S_2(t)\left[d\widetilde{W}_1(t) + 2\,d\widetilde{W}_2(t)\right], \end{aligned}$$

where $\widetilde{W}_1$ and $\widetilde{W}_2$ are independent Brownian motions. Let $g(x_1, x_2)$ be a bounded payoff function. Consider the American option with payoff function g and expiration $T < \infty$. Define,

$$\begin{aligned} &v(t, x_1, x_2) \\ &\quad = \sup\left\{ \tilde{E}[e^{-r\tau} g(S_1(\tau), S_2(\tau)) \Big| S_1(t)=x_1, S_2(t)=x_2 \Big];\ \tau \text{ is a stopping time, } t \leq \tau \leq T. \right\} \end{aligned}$$

Thus $v(t, S_1(t), S_2(t))$ is the price of the American option at time t, assuming it has not been exercised yet.

Find the linear complementarity equations for $v(t, x_1, x_2)$. (Note, the time parameter t is a factor, as in the American option with finite expiration.) Hint: first set up the martingale and supermartingale conditions for characterizing v, by generalizing Theorem 3 of the *Notes to Lecture 8*.

3. For background on this problem, which is a review of multidimensional market modeling as summarized in section 5.4.2 of Shreve, see section 2 of the *Notes to Lectures 9*. For part (b), review the multi-dimensional Girsanov theorem, section 5.4.1 of Shreve. This is also reviewed on Section 4 of the *Notes to Lectures 9*.

(a) Consider a market with three risky assets, whose prices in dollars are $S_1(t)$ and $S_2(t)$ and $Q(t)$.

Let μ_1, μ_2, γ, σ_1, σ_2, and σ_3 be given constants. Write down a model in the form of (5.4.6) in Shreve using a 3-dimensional Brownian motion W $(d = 3)$ so that the model satisfies the following informally given conditions—see Lecture Notes 7:

$$E[\frac{dS_1(t)}{S_1(t)} \mid \mathcal{F}(t)] = \mu_1\, dt \qquad E[\frac{dS_2(t)}{S_2(t)} \mid \mathcal{F}(t)] = \mu_2\, dt, \qquad E[\frac{dQ(t)}{Q(t)} \mid \mathcal{F}(t)] = \gamma\, dt$$

$$\mathrm{Var}\left(\frac{dS_1(t)}{S_1(t)} \mid \mathcal{F}(t)\right) = \sigma_1^2\, dt, \quad \mathrm{Var}\left(\frac{dS_2(t)}{S_2(t)} \mid \mathcal{F}(t)\right) = \sigma_2^2\, dt, \quad \mathrm{Var}\left(\frac{dQ(t)}{Q(t)} \mid \mathcal{F}(t)\right) = \sigma_3^2\, dt$$

$$\mathrm{Cov}\left(\frac{dS_1(t)}{S_1(t)}, \frac{dS_2(t)}{S_2(t)} \mid \mathcal{F}(t)\right) = \frac{1}{4}\sigma_1\sigma_2\, dt, \quad \mathrm{Cov}\left(\frac{dS_1(t)}{S_1(t)}, \frac{dQ(t)}{Q(t)} \mid \mathcal{F}(t)\right) = \frac{1}{2}\sigma_1\sigma_3\, dt,$$

$$\mathrm{Cov}\left(\frac{dS_2(t)}{S_2(t)}, \frac{dQ(t)}{Q(t)} \mid \mathcal{F}(t)\right) = \frac{1}{8}\sigma_2\sigma_3\, dt,$$

Hint: In the notation of (5.4.6), start with $\sigma_{ij} = 0$ for $j > i$.

(b) Assume that σ_1, σ_2 and σ_3 are stricly positive. Assume there is a unique risk-neutral measure $\widetilde{\mathbf{P}}$ where the risk free rate is $R(t)$, an adapted process. Find $\Theta(t) = (\theta_1(t), \theta_2(t), \theta_3(t))$ such that $\widetilde{W}(t) = W(t) + \int_0^t (\theta_1(u), \theta_2(u), \theta_3(u))\, du$ is a Brownian motion under $\widetilde{\mathbf{P}}$.

4. (Extra credit 10pts) Shreve, Exercise 5.8. This exercise should help you understand Theorem 9.2.1 on page 377 when you get to it.

5. (Extra credit 10 pts) Let $S(t)$ denote the price of an asset in a risk-neutral model, and assume,

$$dS(t) = rS(t)\, dt + \sigma S(t)\, d\widetilde{W}(t).$$

Let $M(t)$ be the running maximum of S. Consider an American option with payoff $g(M(t), S(t))$ where g is a bounded function. Let T be the expiration time. The value of the option at time t, assuming it has not yet been exercised, can be written as $v(t, S(t), M(t))$. Find linear complimentarity equations with terminal and boundary conditions for $v(t, x, y)$, and justify why, if you have a function satisfying these conditions, it must indeed be v.